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Lagrange multiplier formalism for spin-2 fields

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Abstract. The Lagrange multiplier method is applied to the description of a spin-2 field. The method bypasses the problem of constraint breakdown and inconsistency of quantisation in the presence of interactions and the theory is shown to remain causal when coupled to an external electromagnetic field. Canonical quantisation of the field is carried out and a covariant propagator obtained. The massless limit of the theory is also discussed.

1. Introduction

The question of anomalies in interacting higher-spin field theories and possible ways of resolving them has attracted considerable attention in recent times. Higher spin theory, from its very inception, has been haunted by troubles of one sort or another. The initial difficulty of algebraic inconsistencies which was noted soon after the formulation of higher-spin wave equations nearly four decades ago was, however, overcome without too much effort by means of a Lagrangian formulation by Pauli and Fierz (1939). But the surprising discovery made by Johnson and Sudarshan (1961) that it is impossible to quantise consistently a spin-$\frac{3}{2}$ field coupled to an external electromagnetic field rekindled interest in the maladies inherent in higher-spin field theories. Velo and Zwanziger (1969a, b) subsequently demonstrated that troubles are present at the classical level itself and that, considered as classical wave equations, the equations of interacting higher-spin fields admit solutions that either do not propagate or propagate at a speed exceeding that of light. Extensive investigations by a number of workers, following the pioneering work of Velo and Zwanziger, have brought to light a variety of abnormalities afflicting higher-spin field theory at the classical as well as at the quantised level. Breakdown of constraint relations (Velo and Zwanziger 1969b, Hagen 1971, Nath *et al* 1971, Jenkins 1974), onset of Lorentz non-covariance (Jenkins 1971a,b, Babu Joseph and Sabir 1976, Mathews *et al* 1976), appearance of imaginary energy eigenvalues (Goldman and Tsai 1971a, b, Tsai and Yildiz 1971, Seetharaman *et al* 1975, Mathews *et al* 1976) and instability in the sense of Wightman (1968, 1976) are, apart from causality violation, forms of pathologies from which higher spin theories suffer.

While over the years more and more maladies have been exposed, and examples of ailing theories have multiplied, there is as yet no definite understanding as to the failure of field theory to give a consistent account of higher spin particles and their interactions. These range from minor modifications of the Lagrangians to alternate formulations without subsidiary conditions and the inclusion of gravitation and supersymmetry. The interest in the Bhabha first-order equations (Bhabha 1945, 1949) has been revived in

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this context. Though they remain causal when interactions are introduced, the quantisation of Bhabha fields needs an indefinite metric and there is no known way to eliminate the negative norm states from the theory (Krajcik and Nieto 1974, 1975a, b, 1976a, b, 1977a, b). On the other hand the Hurley equations (Hurley 1971, 1972, 1974), which also preserve causality, founder on the floor of instability and the physical interpretation of the theory fails. The Bhabha–Gupta (Bhabha 1952, Gupta 1954) equation and the Fisk–Tait (Fisk and Tait 1973) equation provide two examples of theories which have constraint conditions but are free of acausality. Prabhakaran *et al* (1975a, b, 1977) have shown that in both of these theories the total charge is indefinite. It appears that, within the conventional framework, only a multi-mass, multi-spin formulation with an indefinite metric can avoid troubles like causality violation. While the entirely novel approach of supersymmetric theories has had some success in yielding a consistent theory for a spin-$\frac{3}{2}$ field (Deser and Zumino 1976, Freedman 1977, Freedman and Van Nieuwenhuizen 1976, Ferrara and Van Nieuwenhuizen 1976), many problems remain to be settled and it is not immediately obvious why supersymmetry, if needed at all, is essential only for higher spin fields and not for spin-0 or spin-$\frac{1}{2}$ fields which may be formulated without invoking it.

Recently we have suggested a new approach to pathology-free higher-spin field theory by introducing a Lagrange multiplier formalism (Babu Joseph and Sabir 1977). In the Lagrangian formulation, as laid down by Pauli and Fierz (1939), the subsidiary conditions required for the elimination of redundant components are derived along with the equations of motion by the variation of a Lagrangian. Introduction of interactions, as a rule, modifies these constraint relations and this, it may be said, is responsible in several instances for the ensuing anomalies. It was suggested by us that if the existence of constraint conditions is taken into account from the very outset and if these conditions are kept separate from the equations of motion by means of a Lagrange multiplier, many of the usual pathologies of higher spin theories may be overcome. The method was developed with the example of a spin-$\frac{3}{2}$ field, and the absence of acausality and imaginary energy eigenvalues for electromagnetic interaction was demonstrated. Quantisation was carried out in an indefinite metric space and for minimal coupling a unitary S-matrix was constructed by introducing a fictitious particle and an additional vertex.

In the present work the Lagrange multiplier formalism is extended to the description of a spin-2 field. Just as in the spin-$\frac{3}{2}$ case, additional ghost particles are present besides the spin-2 particle and the field is quantised canonically with an indefinite metric. This method bypasses the problems of constraint breakdown and inconsistency in quantisation in the presence of interactions and the theory is shown to remain causal when coupled to an external electromagnetic field. Pathologies apart, another noteworthy feature of the present approach is that it allows a natural discussion of the massless case of the theory by taking the limit $m \to 0$. A discussion is given of the physically interesting case of the massless limit and the true propagator is derived.

Section 2 sets forth the basic formulation of the Lagrange multiplier method as applied to a spin-2 field. Canonical quantisation of the field is carried out in § 3 where the consistency of the procedure is also demonstrated and the propagators are obtained. Minimal coupling is considered in § 4 and it is shown that field propagation remains causal in the presence of an external electromagnetic field. Section 5 contains a discussion of the massless limit and § 6 sums up the main results.

2. Formulation

An irreducible spin-2 field of mass m is described by the set of equations

$$(\Box - m^2)\psi_{\mu\nu} = 0 \tag{1}$$

$$\partial_\mu \psi_{\mu\nu} = 0 \tag{2}$$

$$\psi_{\mu\mu} = 0 \tag{3}$$

where $\psi_{\mu\nu}$ is a symmetric tensor of rank two. A general Lagrangian density $\mathscr{L}_0$ given by

$$\mathscr{L}_0 = -\partial_\lambda \psi^\dagger_{\mu\nu}\partial_\lambda\psi_{\mu\nu} - m^2\psi^\dagger_{\mu\nu}\psi_{\mu\nu} + \partial_\lambda\psi^\dagger_{\mu\nu}\partial_\mu\psi_{\lambda\nu} + \partial_\mu\psi^\dagger_{\mu\nu}\partial_\lambda\psi_{\lambda\nu}$$
$$+ A(\partial_\mu\psi^\dagger_{\mu\nu}\,\partial_\nu\psi + \partial_\nu\psi^\dagger\partial_\mu\psi_{\mu\nu}) + B\,\partial_\nu\psi^\dagger\partial_\nu\psi + m^2C\psi^\dagger\psi, \tag{4}$$

where $\psi = \psi_{\mu\mu}$ and A, B, C are arbitrary parameters, will yield the equations (1)–(3) where A, B, C are restricted by the conditions

$$A = \tfrac{1}{2}, \qquad B = \tfrac{3}{2}A^2 + A + \tfrac{1}{2}, \qquad C = 3A^2 + A + 1. \tag{5}$$

This is the formulation as given by Nath (1965), and under the stipulated conditions the Lagrangian (4) is equivalent to the Fierz–Pauli Lagrangian for the spin-2 field. In the presence of minimal electromagnetic coupling the Nath and Fierz–Pauli formulations suffer from acausality of propagation and inconsistency of quantisation.

In the Lagrange multiplier method we take $\mathscr{L}_0$ as the Lagrangian to start with. It is now assumed that a constraint condition of the form (2) holds between the components of $\psi_{\mu\nu}$, and this is incorporated into the Lagrangian by means of a multiplier field $\chi_\mu(x)$ which for reasons of relativistic covariance must be a four-vector. We take

$$\mathscr{L} = \mathscr{L}_0 + \chi^\dagger_\nu\partial_\mu\psi_{\mu\nu} + \chi_\nu\partial_\mu\psi^\dagger_{\mu\nu}. \tag{6}$$

Here it may be noted that only one of the set of constraint relations (2) and (3) has been taken into account; a discussion of the formalism with both sets of constraints is given in appendix 1.

Variation of $\psi^\dagger_{\mu\nu}$ in the Lagrangian (6) yields the equation of motion

$$(\Box - m^2)\psi_{\mu\nu} - (\partial_\lambda\partial_\mu\psi_{\lambda\nu} + \partial_\lambda\partial_\nu\psi_{\lambda\mu}) - A\partial_\mu\partial_\nu\psi$$
$$- \delta_{\mu\nu}(A\partial_\rho\partial_\lambda\psi_{\rho\lambda} + B\Box\psi) + \delta_{\mu\nu}m^2C\psi = \tfrac{1}{2}(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu) \tag{7}$$

and the variation of $\chi^\dagger_\mu$ reproduces the constraint (2). Making use of equation (2) in equation (7), it follows that

$$(\Box - m^2)\psi_{\mu\nu} - A\partial_\mu\partial_\nu\psi - \delta_{\mu\nu}B\Box\psi + \delta_{\mu\nu}m^2C\psi = \tfrac{1}{2}(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu). \tag{8}$$

Putting $\mu = \nu$ and summing over μ in the above equation, we have

$$(1 - A - 4B)\Box\psi + (4C - 1)m^2\psi = \partial_\mu\chi_\mu. \tag{9}$$

Taking the divergence of equation (8) there results, when account is taken of condition (2),

$$-(A + B)\Box\partial_\nu\psi + Cm^2\partial_\nu\psi = \tfrac{1}{2}(\Box\chi_\nu + \partial_\nu\partial_\mu\chi_\mu). \tag{10}$$

Combining equations (9) and (10), we arrive at the relation

$$\Box\chi_\nu = (2B - A - 1)\Box\partial_\nu\psi - (2C - 1)m^2\partial_\nu\psi. \tag{11}$$

In contrast to the situation we encountered in the spin-$\frac{3}{2}$ case where the constraint relation for the spin-$\frac{3}{2}$ field was purely algebraic, it appears that in the present case, because of the derivative nature of the constraint, the multiplier field χ_μ cannot be expressed locally in terms of the components of $\psi_{\mu\nu}$. However, the multiplier field can be eliminated from equation (8) by operating with $\Box$ on either side of the equation and making use of equation (11).

$$\Box(\Box - m^2)\psi_{\mu\nu} + (1-2B)\Box\partial_\mu\partial_\nu\psi - \delta_{\mu\nu}B(\Box)^2\psi + \delta_{\mu\nu}Cm^2\Box\psi + (2C-1)m^2\partial_\mu\partial_\nu\psi = 0. \tag{12}$$

If it is now assumed that $B = C \neq \frac{1}{3}$, it is immediately seen from equation (12) that

$$\Box(\Box - m^2)\psi = 0 \tag{13}$$

and as a consequence

$$(\Box)^2(\Box - m^2)\psi_{\mu\nu} = 0. \tag{14}$$

The symmetric tensor field in the present theory possesses six independent components. In addition, a vector field is present and its divergence is related to ψ through equation (9). Hence it may be inferred that the present formulation would contain a spin-1 and spin-0 particle in addition to the spin-2 one. This conclusion is further borne out by a study of the quantised version of the above formalism.

3. Quantisation

Since in the Lagrangian we consider the variations of all the components to be independent of one another, the canonical quantisation procedure may be adopted for deriving the field commutation relations. The canonical momenta $\pi_{\mu\nu}$ and $\overset{\dagger}{\pi}_{\mu\nu}$ are defined by

$$\begin{aligned}\pi_{\mu\nu} = \dot{\psi}^{\dagger}_{\mu\nu} &- \tfrac{1}{2}\mathrm{i}(\partial_\mu\psi^{\dagger}_{4\nu} + \partial_\nu\psi^{\dagger}_{4\mu}) - \tfrac{1}{2}\mathrm{i}(\delta_{\mu4}\partial_\rho\psi^{\dagger}_{\rho\nu} + \delta_{\nu4}\partial_\rho\psi^{\dagger}_{\rho\mu}) \\ &- \tfrac{1}{2}\mathrm{i}A(\delta_{\mu4}\partial_\nu\psi^{\dagger} + \delta_{\nu4}\partial_\mu\psi^{\dagger}) - B\dot{\psi}^{\dagger}\delta_{\mu\nu} \\ &- \tfrac{1}{2}\mathrm{i}(\delta_{\mu4}\chi^{\dagger}_{\nu} + \delta_{\nu4}\chi^{\dagger}_{\mu}) - \mathrm{i}A\delta_{\mu\nu}\partial_\rho\psi^{\dagger}_{\rho4},\end{aligned} \tag{15a}$$

$$\begin{aligned}\overset{\dagger}{\pi}_{\mu\nu} = \dot{\psi}_{\mu\nu} &- \tfrac{1}{2}\mathrm{i}(\partial_\mu\psi_{4\nu} + \partial_\nu\psi_{4\mu}) - \tfrac{1}{2}\mathrm{i}(\delta_{\mu4}\partial_\rho\psi_{\rho\nu} + \delta_{\nu4}\partial_\rho\psi_{\rho\mu}) \\ &- \tfrac{1}{2}\mathrm{i}A(\delta_{\mu4}\partial_\nu\psi + \delta_{\nu4}\partial_\mu\psi) - B\delta_{\mu\nu}\dot{\psi} \\ &- \tfrac{1}{2}\mathrm{i}(\delta_{\mu4}\chi_\nu + \delta_{\nu4}\chi_\mu) - \mathrm{i}A\delta_{\mu\nu}\partial_\rho\psi_{\rho4}.\end{aligned} \tag{15b}$$

The Hamiltonian density of the field is easily evaluated with the aid of equations (15), (6), (7) and (2):

$$\begin{aligned}\mathscr{H} = \Big\{ &(\overset{\dagger}{\pi}_{ij} + \mathrm{i}\partial_i\psi_{4j})(\pi_{ij} + \mathrm{i}\partial_i\psi^{\dagger}_{4j}) + \frac{B}{1-3B}[\overset{\dagger}{\pi}_{kk}\pi_{jj} + \tfrac{1}{2}\mathrm{i}(\overset{\dagger}{\pi}_{kk}\partial_j\psi^{\dagger}_{4j} + \pi_{kk}\partial_j\psi_{4j})] \\ &+ \partial_j\psi_{ij}(\partial_k\psi^{\dagger}_{ik} + \partial_i\psi^{\dagger}_{44} - 2\mathrm{i}\pi_{i4}) \\ &- \mathrm{i}\partial_j\psi_{4j}\left(\pi_{44} + \frac{B}{2(1-3B)}\pi_{kk}\right) + \partial_j\psi^{\dagger}_{ij}(\partial_k\psi_{ik} + \partial_i\psi_{44} - 2\overset{\dagger}{\pi}_{i4}) \\ &- \mathrm{i}\partial_j\psi^{\dagger}_{4j}\left(\overset{\dagger}{\pi}_{44} + \frac{B}{2(1-3B)}\overset{\dagger}{\pi}_{kk}\right) - \partial_j\psi^{\dagger}_{j\nu}\partial_j\psi_{i\nu} \\ &+ \partial_i\psi^{\dagger}_{\mu\nu}\partial_i\psi_{\mu\nu} + m^2\psi^{\dagger}_{\mu\nu}\psi_{\mu\nu} - B\partial_i\psi^{\dagger}\partial_i\psi - m^2C\psi^{\dagger}\psi\Big\}.\end{aligned} \tag{16}$$

As might have been expected, it is seen from the above expression that the Hamiltonian is not positive-definite, due to the presence of additional spin constituents, and hence quantisation will have to be carried out in an indefinite metric space. Taking into consideration the symmetry of the field, the canonical commutation relations may be written in the form

$$[\psi_{\mu\nu}(x),\ \pi_{\rho\lambda}(x')]_{x_0=x_0'}=\tfrac{1}{2}\mathrm{i}(\delta_{\mu\rho}\delta_{\nu\lambda}+\delta_{\mu\lambda}\delta_{\nu\rho})\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{17a}$$

$$[\psi^{\dagger}_{\mu\nu}(x),\ \pi^{\dagger}_{\rho\lambda}(x')]_{x_0=x_0'}=\tfrac{1}{2}\mathrm{i}(\delta_{\mu\rho}\delta_{\nu\lambda}+\delta_{\mu\lambda}\delta_{\nu\rho})\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{17b}$$

$$[\psi_{\mu\nu}(x),\ \psi_{\rho\lambda}(x')]_{x_0=x_0'}=[\psi_{\mu\nu}(x),\ \psi^{\dagger}_{\rho\lambda}(x')]=0 \tag{17c}$$

$$[\pi_{\mu\nu}(x),\ \pi_{\rho\lambda}(x')]_{x_0=x_0'}=[\pi_{\mu\nu}(x),\ \pi^{\dagger}_{\rho\lambda}(x')]=0. \tag{17d}$$

These commutation relations are not automatically consistent with the constraint relation (2). By requiring that (17) be compatible with the constraint condition, we can re-express these commutation relations in terms of $\psi_{\mu\nu}$, $\dot{\psi}_{\mu\nu}$, $\psi^{\dagger}_{\mu\nu}$, $\dot{\psi}^{\dagger}_{\mu\nu}$, χ_{μ} and $\chi^{\dagger}_{\mu}$ with the aid of the definitions (15). The full set of commutation relations is presented in appendix 2.

We shall now demonstrate the consistency of the quantisation procedure by showing that the Heisenberg equations

$$\dot{\psi}_{\mu\nu}(x)=(1/\mathrm{i})[\psi_{\mu\nu}(x),H] \tag{18a}$$

$$\dot{\pi}_{\mu\nu}(x)=(1/\mathrm{i})[\pi_{\mu\nu}(x),H], \tag{18b}$$

where $H=\int\mathscr{H}\,\mathrm{d}^3x$ and $\mathscr{H}$ is given by (16), reproduce the equations (8) and (2) when the canonical commutations in appendix 2 are employed in (18). To verify this, consider first the equations for ψ_{i4} and ψ_{44} which by equation (18) are

$$\dot{\psi}_{i4}=(1/\mathrm{i})[\psi_{i4},H]=-\mathrm{i}\partial_j\psi_{ij} \tag{19a}$$

$$\dot{\psi}_{44}=(1/\mathrm{i})[\psi_{44},H]=-\partial_j\psi_{4j}. \tag{19b}$$

These two equations together are identical with the constraint condition (2). Consider now the equations for the components ψ_{ij} and $\pi^{\dagger}_{ij}$,

$$\dot{\psi}_{ij}=(1/\mathrm{i})[\psi_{ij},H]=\pi^{\dagger}_{ij}+\tfrac{1}{2}\mathrm{i}(\partial_i\psi_{4j}+\partial_j\psi_{4i})+\frac{B}{1-3B}\delta_{ij}\pi^{\dagger}_{kk} \tag{20}$$

$$\begin{aligned}\dot{\pi}^{\dagger}_{ij}&=(1/\mathrm{i})[\pi^{\dagger}_{ij},H]\\&=-\mathrm{i}(\partial_i\pi^{\dagger}_{j4}+\partial_j\pi^{\dagger}_{i4})+\partial_i\partial_j\psi_{44}+\partial^2_k\psi_{ij}-m^2\psi_{ij}-B\delta_{ij}\partial^2_k\psi+m^2(\delta_{ij}\psi).\end{aligned} \tag{21}$$

Equation (20) is an identity corresponding to one of the relations in the definition of canonical momenta (15). Taking the time derivative of equation (20) and substituting from equation (21) and making use of the already derived equation (19) and the defining relation (15a) there emerges the following second-order equation:

$$(\Box-m^2)\psi_{ij}-A\partial_i\partial_j\psi-B\delta_{ij}\Box\psi+\delta_{ij}m^2C\psi=\tfrac{1}{2}(\partial_i\chi_j+\partial_j\chi_i) \tag{22}$$

which is identical with the ij components of equation (8). The rest of the equations of motion are recovered in a similar fashion by considering the Heisenberg equations for $\pi^{\dagger}_{i4}$ and π_{44}.

The general form of the commutator $[\psi_{\mu\nu}(x), \psi^{\dagger}_{\rho\lambda}(x')]$ for arbitrary separations can be inferred by invoking the principles of relativistic covariance and local commutativity and equations (14) and (2):

$$[\psi_{\mu\nu}(x), \psi^{\dagger}_{\rho\lambda}(x')]$$
$$= a[\tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\rho}) - \delta_{\mu\nu}\delta_{\rho\lambda} + m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu)$$
$$-\tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda + \delta_{\mu\lambda}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)]\Delta(x-x'; m^2)$$
$$+ b[m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu) - \tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda + \delta_{\mu\lambda}\partial_\nu\partial_\rho$$
$$+ \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)]\Delta(x-x'; 0). \tag{23}$$

Apart from the constants a and b the coefficients of the various terms in equation (23) have been so chosen that the constraint relation (2) is consistent with the general commutator. If the coefficients a and b are assigned the values $a = \mathrm{i}$, $b = -\mathrm{i}$ the equal time commutation relations (appendix 2) can be recovered from equation (24) for the choice of the parameter $B = \frac{1}{2}$. In a similar manner we can write down the following general commutation relations involving χ_μ:

$$[\psi_{\mu\nu}(x), \chi^{\dagger}_{\rho}(x')] = -\mathrm{i}(\delta_{\mu\rho}\partial_\nu + \delta_{\nu\rho}\partial_\mu - \delta_{\mu\nu}\partial_\rho)\Delta(x-x'; 0) \tag{24}$$

$$[\chi_\mu(x), \chi_\nu(x')] = -\mathrm{i}(\delta_{\mu\nu} - \partial_\mu\partial_\nu)\Delta(x-x'; 0). \tag{25}$$

These commutation relations will be consistent with the equal time commutators given in appendix 2 provided the parameter A is set equal to zero.

The Feynman propagator of the field is evaluated in the customary way. In this case the normal-dependent terms drop off and the propagator is rigorously given by

$$\Delta_{\mathrm{F}\,\mu\nu,\rho\lambda}(x-x')$$
$$= \langle 0|T\psi_{\mu\nu}(x)\psi^{\dagger}_{\rho\lambda}(x')|0\rangle$$
$$= [\tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\rho}) - \delta_{\mu\nu}\delta_{\rho\lambda} + m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu)$$
$$+\tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda + \delta_{\mu\lambda}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)]\Delta_{\mathrm{F}}(x-x'; m^2)$$
$$-[m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu) - \tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda + \delta_{\mu\lambda}\partial_\nu\partial_\rho$$
$$+ \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)]\Delta_{\mathrm{F}}(x-x'; 0). \tag{26}$$

This may be rewritten in the form

$$\Delta_{\mathrm{F}\,\mu\nu,\rho\lambda}(x-x') = \Delta^{(2)}_{\mathrm{F}\,\mu\nu,\rho\lambda}(x-x') - \tfrac{2}{3}(\delta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu)(\delta_{\rho\lambda} - m^{-2}\partial_\rho\partial_\lambda)\Delta_{\mathrm{F}}(x-x'; m^2)$$
$$-[m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu) - \tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda$$
$$+ \delta_{\mu\lambda}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)]\Delta_{\mathrm{F}}(x-x'; 0) \tag{27}$$

where $\Delta^{(2)}_{\mathrm{F}}$ is the usual massive spin-2 propagator

$$\Delta^{(2)}_{\mathrm{F}\,\mu\nu,\rho\lambda}(x-x') = [\tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\rho}) - \tfrac{1}{3}\delta_{\mu\nu}\delta_{\rho\lambda}$$
$$+\tfrac{1}{3}m^{-2}(\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu) - \tfrac{1}{2}m^{-2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda + \delta_{\mu\lambda}\partial_\nu\partial_\rho$$
$$+ \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho) + \tfrac{2}{3}m^{-4}\partial_\mu\partial_\nu\partial_\rho\partial_\lambda]\Delta_{\mathrm{F}}(x-x'; m^2). \tag{28}$$

The other propagators are given by

$$\langle 0|T\psi_{\mu\nu}(x)\chi^{\dagger}_{\rho}(x')|0\rangle = -(\delta_{\mu\rho}\partial_\nu + \delta_{\nu\rho}\partial_\mu - \delta_{\mu\nu}\partial_\rho)\Delta_{\mathrm{F}}(x-x'; 0) \tag{29}$$

$$\langle 0|T\chi_\mu(x)\chi^{\dagger}_{\nu}(x')|0\rangle = -(\delta_{\mu\nu} - \partial_\mu\partial_\nu)\Delta_{\mathrm{F}}(x-x') - \mathrm{i}\delta_{\mu 4}\delta_{\nu 4}\delta^{(4)}(x-x'). \tag{30}$$

Since $\chi_\mu(x)$ represents a negative-metric-carrying ghost particle which will not be present in the physical states, the above propagators will not be of further use and the non-covariant form of equation (30) is of no consequence.

4. Electromagnetic interaction

We shall now consider the interaction of the above-described symmetric tensor field with an external electromagnetic field. The questions of constraint loss and the Johnson–Sudarshan type of inconsistency on quantisation do not arise at all in the present framework. We now show that the present formulation is free of the trouble of acausality also.

Introducing minimal coupling into the Lagrangian (6) by the replacement $\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ the equations of motion resulting from the variations of $\psi^\dagger_{\mu\nu}$ and $\chi^\dagger_\nu$ become

$$\begin{aligned}(D^2-m^2)\psi_{\mu\nu} &- (D_\lambda D_\mu \psi_{\lambda\nu} + D_\lambda D_\nu \psi_{\lambda\mu}) - AD_\mu D_\nu \psi \\ &- \delta_{\mu\nu}[D_\rho D_\lambda \psi_{\rho\lambda} + BD^2\psi] + \delta_{\mu\nu} C m^2 \psi \\ &= \tfrac{1}{2}(D_\mu \chi_\nu + D_\nu \chi_\mu) \qquad (31)\end{aligned}$$

$$D_\mu \psi_{\mu\nu} = 0. \qquad (32)$$

It may be noted here that, as contrasted with the situation of the spin-$\frac{3}{2}$ theory developed in Babu Joseph and Sabir (1977) where the constraint condition was free of derivatives and was unaffected by the interaction, in the present case, the requirement of gauge invariance leads to a minimal modification of the constraint (2). Substituting (32) in (31), with the aid of the relation

$$[D_\mu, D_\nu] = \mathrm{i}eF_{\mu\nu}, \qquad (33)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we find

$$\begin{aligned}(D^2-m^2)\psi_{\mu\nu} &- AD_\mu D_\nu \psi - \delta_{\mu\nu} BD^2 \psi + \delta_{\mu\nu} m^2 C\psi - \mathrm{i}e(F_{\lambda\mu}\psi_{\lambda\nu} + F_{\lambda\nu}\psi_{\lambda\mu}) \\ &= \tfrac{1}{2}(D_\mu \chi_\nu + D_\nu \chi_\mu). \qquad (34)\end{aligned}$$

Contracting the indices μ and ν, taking the divergence of equation (34), and then combining the two resulting relations, it is found that the equation obeyed by χ_ν is

$$D^2\chi_\nu = -AD^2D_\nu\psi - BD_\nu D^2\psi + m^2CD_\nu\psi - \mathrm{i}e(F_{\mu\lambda}D_\lambda\psi_{\mu\nu} - F_{\lambda\nu}D_\mu\psi_{\lambda\nu}). \qquad (35)$$

This relation may be used for eliminating from equation (31), and we obtain the resulting equation as

$$D^2(D^2-m^2)\psi_{\mu\nu} + BD_\mu D_\nu D^2\psi - \delta_{\mu\nu}B(D^2)^2\psi + \text{lower derivative terms} = 0. \qquad (36)$$

We have not shown the lower derivative terms explicitly because these are not important in determining the nature of field propagation, which depends solely on the highest-order derivatives. The characteristic determinant of the system of equations (36) is evaluated with the result

$$D(n) = (1-B)^3(n^4)^{10}. \qquad (37)$$

Since $B \neq 1$ in the present theory (the choice $B = \frac{1}{2}$ having been made for consistent quantisation), the characteristic determinant will not vanish identically and, setting

$D(n)=0$, it is evident that the propagation of the interacting field is light-like and hence causal.

5. Massless limit

Another merit of the present formulation is that the limit $m \to 0$ can be taken smoothly without encountering any difficulty. The equation of motion in this case is

$$\Box\psi_{\mu\nu} - A\partial_\mu\partial_\nu\psi - \delta_{\mu\nu}B\Box\psi = \tfrac{1}{2}(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu) \tag{38}$$

but the constraint remains the same as equation (2). The formalism can be developed in a way exactly analogous to the massive case. It suffices to note that in the massless case equation (38) and equation (2) are invariant under the gauge transformation

$$\psi_{\mu\nu}(x) \to \psi_{\mu\nu}(x) + \tfrac{1}{2}[\partial_\mu\xi_\nu(x) + \partial_\nu\xi_\mu(x)] \tag{39}$$

$$\chi_\mu(x) \to \chi_\mu(x) \tag{40}$$

provided the gauge functions ξ_μ obey the conditions

$$\Box\xi_\mu(x) = 0 \tag{41}$$

$$\partial_\mu\xi_\mu(x) = 0. \tag{42}$$

This gauge freedom serves to reduce the number of degrees of freedom in the massless case.

The Feynman propagator of the massless field is obtained by letting $m \to 0$ in the expression (26):

$$\begin{aligned} D_{\mathrm{F}\ \mu\nu,\rho\lambda}(x-x') &= [\tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\rho}) - \delta_{\mu\nu}\delta_{\rho\lambda}] \\ &\quad \times \Delta_{\mathrm{F}}(x-x';0) + (\delta_{\mu\nu}\partial_\rho\partial_\lambda + \delta_{\rho\lambda}\partial_\mu\partial_\nu - \tfrac{1}{2}(\delta_{\mu\rho}\partial_\nu\partial_\lambda \\ &\quad + \delta_{\mu\lambda}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\lambda + \delta_{\nu\lambda}\partial_\mu\partial_\rho)E(x-x') \end{aligned} \tag{43}$$

where

$$E(x-x') = \lim_{m^2\to 0} \frac{\partial}{\partial m^2}\Delta_{\mathrm{F}}(x-x';m^2).$$

The expansion

$$\Delta_{\mathrm{F}}(x-x';m^2) = \Delta_{\mathrm{F}}(x-x';0) + m^2\frac{\partial}{\partial m^2}\Delta_{\mathrm{F}}(x-x';m^2)|_{m^2=0} + \dots \tag{44}$$

was used in deriving equation (43). In the expression (43) the term $[\frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\rho}) - \delta_{\mu\nu}\delta_{\rho\lambda}]\Delta_{\mathrm{F}}(x-x';0)$ corresponds to the actual propagator of the massless spin-2 field while the other term represents the ghost particles. One noteworthy aspect of the present approach is that the true massless spin-2 propagator is obtained simply by taking the limit $m \to 0$ in equation (26) whereas this does not happen in the conventional formulations even if it is assumed that the derivative terms can be left out.

6. Conclusion

The Lagrange multiplier formalism developed above for the spin-2 field has the attractive feature that pathologies like constraint loss, inconsistency in quantisation and acausal propagation do not make their appearance in this approach. However, this is achieved at the cost of introducing an indefinite metric theory and this brings its own share of troubles. In the spin-$\frac{3}{2}$ case (Babu Joseph and Sabir 1977) with minimal electromagnetic interaction introduced, it was found that the ghost particle also gets coupled to the electromagnetic field and this leads to the non-unitarity of the S-matrix. A procedure was outlined there for the removal of non-unitarity by introducing a fictitious particle and an additional vertex. As is evident from an inspection of equation (35), the non-unitarity problem is bound to arise in the present example as well when the S-matrix is constructed. But, on account of the difficulty of isolating the negative-metric carrying components, the method followed in the spin-$\frac{3}{2}$ case for the restoration of unitarity is not easily carried out in this instance. We hope, however, that the procedure of introducing additional vertices and fictitious particles would resolve the difficulty in the present case as well.

The possibility of discussing the massless limit in a straightforward way is another advantage of the present approach and this may be used as a springboard for the formulation of a new gravitational theory involving tensor and additional ghost particles. This problem is currently under investigation.

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Appendix 1

A Lagrange multiplier formalism can be developed by incorporating both sets of constraints (2) and (3). We take the Lagrangian in this case to be

$$\mathscr{L} = \mathscr{L}_0 + \chi^\dagger_\nu \partial_\mu \psi_{\mu\nu} + \chi_\nu \partial_\mu \psi^\dagger_{\mu\nu} + \xi^\dagger \psi + \xi \psi^\dagger \tag{A1.1}$$

where ξ is a scalar multiplier field. The equation of motion is

$$(\Box - m^2)\psi_{\mu\nu} = \tfrac{1}{2}(\partial_\mu \chi_\nu + \partial_\nu \chi_\mu) - \delta_{\mu\nu}\xi. \tag{A1.2}$$

Taking into account the constraints (2) and (3), it follows from (A1.2) that

$$\partial_\mu \chi_\mu - 4\xi = 0 \tag{A1.3}$$

$$\Box \chi_\nu + \partial_\nu \partial_\mu \chi_\mu - 2\partial_\nu \xi = 0. \tag{A1.4}$$

Equation (A1.3) expresses the scalar multiplier field ξ in terms of $\partial_\mu \chi_\mu$. By combining equations (A1.3) and (A1.4) the equation obeyed by χ_μ can be derived:

$$(\Box)^2 \chi_\nu = 0. \tag{A1.5}$$

From equations (A1.5) and (A1.2) it follows further that

$$(\Box)^2(\Box - m^2)\psi_{\mu\nu} = 0 \tag{A1.6}$$

which is the same as equation (14).

The quantisation procedure can be developed along the lines in § 3. However, a difficulty that appears in this case is that the Heisenberg equations reproduce the field equation (A1.2) only if $\partial_\mu \chi_\mu = 0$. Another obstacle is that a general commutator satisfying the constraints (A1.2) and (A1.3) and other requirements, while reproducing the equal time anticommutation relations, cannot be written down. It is on account of these complexities that a single multiplier formalism has been adopted in the text.

Appendix 2

The equal time commutation relations involving the components $\psi_{\mu\nu}$, $\dot{\psi}_{\mu\nu}$, $\psi^{\dagger}_{\mu\nu}$, $\dot{\psi}^{\dagger}_{\mu\nu}$, χ_{μ} and $\chi^{\dagger}_{\mu}$ are as follows:

$$[\psi_{ij}(x), \dot{\psi}^{\dagger}_{kl}(x')] = \mathrm{i}\left[\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{B}{1-3B}\delta_{ij}\delta_{kl}\right]\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.1}$$

$$[\psi_{ij}(x), \dot{\psi}^{\dagger}_{k4}(x')] = [\psi_{ij}(x), \psi^{\dagger}_{44}(x')] = 0 \tag{A2.2}$$

$$[\psi_{i4}(x), \dot{\psi}^{\dagger}_{ij}(x')] = [\psi_{i4}(x), \dot{\psi}^{\dagger}_{44}(x')] = 0 \tag{A2.3}$$

$$[\psi_{44}(x), \dot{\psi}^{\dagger}_{ij}(x')] = [\psi_{44}(x), \dot{\psi}^{\dagger}_{i4}(x')] = 0 \tag{A2.4}$$

$$[\dot{\psi}_{ij}(x), \psi^{\dagger}_{kl}(x')] = 0 \tag{A2.5}$$

$$[\dot{\psi}_{i4}(x), \dot{\psi}^{\dagger}_{kl}(x')] = \left[\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{B}{1-3B}\delta_{ij}\delta_{kl}\right]\partial_j\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.6}$$

$$[\dot{\psi}_{44}(x), \dot{\psi}^{\dagger}_{ij}(x')] = 0 \tag{A2.7}$$

$$[\dot{\psi}_{i4}(x), \dot{\psi}^{\dagger}_{j4}(x')] = [\dot{\psi}_{44}(x), \dot{\psi}^{\dagger}_{44}(x')] = 0 \tag{A2.8}$$

$$[\dot{\psi}_{i4}(x), \dot{\psi}^{\dagger}_{44}(x')] = 0 \tag{A2.9}$$

$$[\psi_{ij}(x), \chi^{\dagger}_{k}(x')] = 0 \tag{A2.10}$$

$$[\psi_{i4}(x), \chi^{\dagger}_{j}(x')] = 0 \tag{A2.11}$$

$$[\psi_{44}(x), \chi^{\dagger}_{j}(x')] = 0 \tag{A2.12}$$

$$[\psi_{ij}(x), \chi^{\dagger}_{4}(x')] = -\frac{(A+B)}{1-3B}\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.13}$$

$$[\psi_{i4}(x), \chi^{\dagger}_{4}(x')] = 0 \tag{A2.14}$$

$$[\dot{\psi}_{ij}(x), \chi^{\dagger}_{k}(x')] = -\mathrm{i}\left[\delta_{ik}\partial_j\delta(\boldsymbol{x}-\boldsymbol{x}')+\delta_{jk}\partial_i\delta(\boldsymbol{x}-\boldsymbol{x}')+\frac{A+B}{1-3B}\delta_{ij}\partial_k\delta(\boldsymbol{x}-\boldsymbol{x}')\right] \tag{A2.15}$$

$$[\psi_{44}(x), \chi^{\dagger}_{4}(x')] = -\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.16}$$

$$[\dot{\psi}_{i4}(x), \chi^{\dagger}_{j}(x')] = 0 \tag{A2.17}$$

$$[\dot{\psi}_{44}(x), \chi^{\dagger}_{i}(x')] = \mathrm{i}\partial_i\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.18}$$

$$[\dot{\psi}_{i4}(x), \chi^{\dagger}_{4}(x')] = \mathrm{i}[(A+B)/(1-3B)]\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{A2.19}$$

$$[\dot{\psi}_{ij}(x), \chi^{\dagger}_{4}(x')] = 0 \tag{A2.20}$$

$$[\dot{\psi}_{44}(x), \chi^{\dagger}_{4}(x')] = 0 \tag{A2.21}$$

$$[\chi_k(x), \chi^{\dagger}_{l}(x')] = 0 \tag{A2.22}$$

$$[\chi_4(x), \chi^{\dagger}_{4}(x')] = 0 \tag{A2.23}$$

$$[\chi_i(x), \chi^{\dagger}_{4}(x')] = -[(1+2A-B)/(1-3B)]\partial_i\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{A2.24}$$

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